③ Rotations of quantum states and operators.

(Euclidean space) $\qquad$ (Hilbert space)

$$R \longrightarrow \mathcal{D}(R)$$

(orthogonal) $\qquad$ (unitary)

A postulate: $\mathcal{D}(R)$ has the same group properties as R.

- Identity: $R \cdot \mathbb{I} = R \Rightarrow \mathcal{D}(R) \cdot \mathbb{1} = \mathcal{D}(R)$
- Closure: $R_1 R_2 = R_3 \Rightarrow \mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_3)$
- Inverse: $RR^{-1} = R^{-1}R = I \Rightarrow \mathcal{D}(R)\mathcal{D}^{-1}(R) = 1$
  $\left(\mathcal{D}^{-1}(R)\mathcal{D}(R) = 1 \right.$
- Associativity: $R_1(R_2R_3) = (R_1R_2)R_3 \equiv R_1R_2R_3$

$$\Rightarrow \mathcal{D}(R_1)\left[\mathcal{D}(R_2)\mathcal{D}(R_3)\right] = \left[\mathcal{D}(R_1)\mathcal{D}(R_2)\right]\mathcal{D}(R_3)$$
$$\equiv \mathcal{D}(R_1)\mathcal{D}(R_2)\mathcal{D}(R_3)$$

- Infinitesimal transformation $\vec{x} \rightarrow \vec{x}' = \vec{x} + \vec{\theta} \times \vec{x}$ $\quad \| \ \vec{\theta} = \theta\hat{n}$

has the same form in R and $\mathcal{D}(R)$ as $\quad (\theta \ll 1)$

$$\mathcal{D}(R) \simeq 1 - \frac{i}{\hbar}\theta_k J_k$$

|| summation Σ_k is omitted for repeated index k

- We will use this notation for all repeated indices.

Task: $\boxed{\text{☆ DEFINE "}J_k\text{" !}}$

But we don't know what's J_k, yet.

- Rotation of a quantum state

$$|\alpha_R\rangle = \mathcal{D}(R)|\alpha\rangle$$

|| NOTE: Here, $\hat{n}$ is fixed.

- Rotation of an operator.

① Scalar operator: $\langle\beta_R|S|\alpha_R\rangle = \langle\beta|\mathcal{D}^\dagger(R)S\mathcal{D}(R)|\alpha\rangle$

has to be invariant. $\qquad = \langle\beta|S|\alpha\rangle.$

$$\Rightarrow \left(1+\frac{i}{\hbar}\theta_k J_k\right) S \left(1-\frac{i}{\hbar}\theta_k J_k\right) = S$$

$$\Rightarrow \quad [\vec{J}, S] = 0 \quad \text{for a scalar operator } S.$$

③ Vector operator $\vec{V} = (V_1, V_2, V_3)^T$ (in 3D)

$$\langle \beta_R | V_i | \alpha_R \rangle = \langle \beta | \mathcal{D}^\dagger(R) V_i \mathcal{D}(R) | \alpha \rangle$$

$$(\text{has to be}) = R_{ij} \langle \beta | V_j | \alpha \rangle$$ (It's just a rotation of a vector.)

$$\Rightarrow \mathcal{D}^\dagger(R) V_i \mathcal{D}(R) = R_{ij} V_j$$

$$V_i + \frac{i}{\hbar}\theta_k [J_k, V_i] = V_i - \varepsilon_{ijk} V_j \theta_k$$

reordering (ijk),

$$\therefore \boxed{[J_i, V_j] = i\hbar\, \varepsilon_{ijk} V_k}$$

$\vec{V}' = \vec{V} + \vec{\theta} \times \vec{V}$

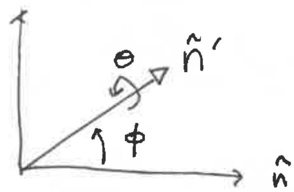
$\rightarrow V_k + \varepsilon_{ijk} \theta_i V_j$

$\rightarrow V_i - \varepsilon_{ijk} V_j \theta_k$.

- Rotation about an rotated axis.

$$\mathcal{D}(R_\phi)\, \mathcal{D}(R_\theta)\, \mathcal{D}^\dagger(R_\phi) \; :$$

$$= \mathcal{D}(R_\phi R_\theta R_\phi^{-1}) = \mathcal{D}(R'_\theta)$$

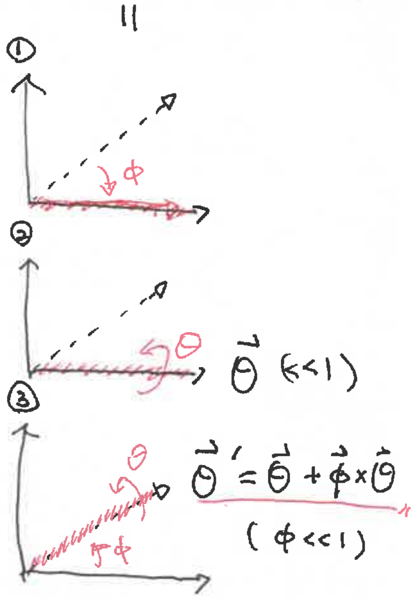


$$\mathcal{D}(R_\phi)\left[1 - \frac{i}{\hbar}\theta_k J_k\right]\mathcal{D}^\dagger(R_\phi)$$

$$= 1 - \frac{i}{\hbar}\theta'_k J_k$$

$$= 1 - \frac{i}{\hbar}\left[\theta_k J_k + \varepsilon_{ijk}\phi_i \theta_j J_k\right]$$

$$= 1 - \frac{i}{\hbar}\theta_k\left[J_k + \varepsilon_{ijk} J_i \phi_j\right]$$



$$\Rightarrow \quad \mathcal{D}(R_\phi)\, J_k\, \mathcal{D}^\dagger(R_\phi) = J_k + \varepsilon_{ijk} J_i \phi_j$$

$$\cancel{J_k} - \frac{i}{\hbar}\cancel{\phi_j}\,[J_j, J_k] = \cancel{J_k} + \varepsilon_{ijk} J_i \cancel{\phi_j}$$

$$\Rightarrow \quad \boxed{[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k}$$

☆☆☆☆☆

Lie algebra, again.

"Fundamental commutation relation" defining the Lie algebra between $\{J_i\}$.

In plain words, you should find $\{J_i\}$ satisfying $[J_i, J_j] = i\hbar\,\varepsilon_{ijk} J_k$

or, you may say, you'll accept any $\{J_i\}$ if it satisfies ↑.

- An obvious choice from C.M.: Angular momentum. (orbital)

rotation: $\quad q_k \to Q_k = q_k + \theta\, \varepsilon_{ijk} n_i q_j$

↓

canonical transformation:

$$\vec{Q} = \vec{q} + \vec{\theta} \times \vec{q}, \quad \vec{\theta} = \theta \hat{n}$$

$$Q_k = q_k + \theta \frac{\partial G}{\partial p_k}$$

$$\Rightarrow \quad G = \varepsilon_{ijk} n_i q_j p_k$$

$$= (\vec{q} \times \vec{p}) \cdot \hat{n} = \vec{L} \cdot \hat{n}$$

∴ Angular momentum is a generator of rotation.

☆ $\Rightarrow \quad \vec{J} = \vec{L}$ from quantum-classical correspondence.

- Q: Are there other possible $\vec{J}$'s ?

A: Yes. There are MANY, as a matter of fact.

ex. spin-$\frac{1}{2}$ operators : S_x, S_y, S_z

$\Rightarrow$ $[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$: we know this!

"orbital" angular momentum : $\vec{L} = \vec{\underset{\sim}{X}} \times \vec{\underset{\sim}{P}} = \vec{J}$

"spin" : all other possibilities, ex. $\vec{J} = \frac{\hbar}{2}\vec{\sigma}$

or, we just say "angular momentum" to call all of them.

(2) Spin-$\frac{1}{2}$ systems and Rotations.

$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$ $\rightarrow$ realization at the Hilbert space dim. = 2.

$\Rightarrow$ $\vec{J} = \vec{S}$ for spin-$\frac{1}{2}$.

H. Dim = 2 $\Leftarrow$ $\{|\uparrow\rangle, |\downarrow\rangle\}$

"spin - $\frac{1}{2}$" $\nLeftarrow$ It'll be shown later...

Wait! We're working on rotations in 3D, aren't we?

But, here it looks like 2D...

$\hookrightarrow$ Rotation is in 3D (x, y, z) spatial coordinates (3D)

so, we're going to rotate $\vec{S} = (S_x, S_y, S_z)^T$,

but now, $S_{x,y,z}$ is not an scalar.

it's a 2×2 matrix. rep. by $\{|\uparrow\rangle, |\downarrow\rangle\}$ - basis (2D)!

We have already seen the rotation of a spin-$\frac{1}{2}$ system, but we didn't just say it's "rotation".

Spin Precession revisited.

: A spin-$\frac{1}{2}$ operator $\vec{S} = (\tilde{S}_x, \tilde{S}_y, \tilde{S}_z)^T$ of an electron time-evolving in a magnetic field.

with the Hamiltonian

$$H = \omega S_z \quad \| \quad \omega = \frac{|e|B}{m_e c}.$$

The time-evolution operator $U(t) = e^{-\frac{i}{\hbar}Ht} = \exp\left[-\frac{i}{\hbar}\tilde{S}_z \omega t\right]$

↳ Heisenberg eq. of motion $\frac{d\vec{S}}{dt} = \frac{1}{i\hbar}[\vec{S}, H] \xrightarrow{\text{sol.}} \vec{S}(t) = U^\dagger(t)\vec{S}\,U(t)$

or the direct time-evolution of a ket $|\alpha, t\rangle = U(t)|\alpha\rangle$.

provide $\quad \langle\alpha| \exp\left[+\frac{i}{\hbar}\tilde{S}_z\omega t\right] \vec{S} \exp\left[-\frac{i}{\hbar}\tilde{S}_z\omega t\right] |\alpha\rangle.$

- There are two ways of computing this.

  1) express $\vec{S}$ in the eigenket basis of $\tilde{S}_z$:

$$S_x = \frac{\hbar}{2}\left[\,|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\,\right]$$
$$S_y = \frac{\hbar}{2}\cdot i\left[\,-|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\,\right]$$
$$S_z = \frac{\hbar}{2}\left[\,|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\,\right]$$

ex. $\quad \frac{\hbar}{2} e^{\frac{i}{\hbar}\tilde{S}_z\omega t}\left[\,|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\,\right] e^{-\frac{i}{\hbar}\tilde{S}_z\omega t}$ for $\tilde{S}_x(t)$.

$$= \frac{\hbar}{2}\left[\, e^{i\frac{\omega t}{2}} |\uparrow\rangle\langle\downarrow| e^{i\frac{\omega t}{2}} + e^{-\frac{i\omega t}{2}} |\downarrow\rangle\langle\uparrow| e^{-\frac{i\omega t}{2}} \,\right]$$

$$= \frac{\hbar}{2}\left[\,\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\cos\omega t + i\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right)\sin\omega t\,\right]$$

$$S_x(t) = S_x\cos\omega t - S_y\sin\omega t.$$

② use $[\tilde{S}_i, \tilde{S}_j] = i\hbar \varepsilon_{ijk} \tilde{S}_k$, only.

2-1. Heisenberg EoM:

$$\frac{d\tilde{S}_x}{dt} = \frac{1}{i\hbar}[\tilde{S}_x, \omega \tilde{S}_z] = -\omega \tilde{S}_y.$$

$$\frac{d\tilde{S}_y}{dt} = \frac{1}{i\hbar}[\tilde{S}_y, \omega \tilde{S}_z] = \omega \tilde{S}_x$$

$$\frac{d\tilde{S}_z}{dt} = \frac{1}{i\hbar}[\tilde{S}_z ; \omega \tilde{S}_z] = 0.$$

$$\Rightarrow \quad \ddot{\tilde{S}}_x = -\omega^2 \tilde{S}_x \rightarrow \tilde{S}_x(t) = \hat{A} e^{i\omega t} + \tilde{B} e^{-i\omega t}$$

$$\underline{\tilde{S}_x(t) = \tilde{S}_x(0)\cos\omega t - \tilde{S}_y(0)\sin\omega t} \leftarrow$$

initial conditions.

$$\tilde{S}_x \equiv \tilde{S}_x(0) = \hat{A} + \tilde{B}$$

$$\tilde{S}_y \equiv \tilde{S}_y(0) = -i(\hat{A} - \tilde{B})$$

2-2. Baker - Campbell - Hausdorff formula (2.3.47) Sakurai.

$$e^{\frac{iS_z\omega t}{\hbar}} S_x e^{-\frac{i}{\hbar}S_z\omega t} = S_x + \frac{i\omega t}{\ominus\cancel{\hbar}}\underbrace{[S_z, S_x]}_{=\cancel{i\hbar} S_y}$$

$$+ \frac{1}{2!}\left(\frac{i\omega t}{\cancel{\hbar}}\right)^2 \underbrace{[S_z, [\quad]]}_{=\cancel{\hbar^2} S_x} + \frac{1}{3!}\left(\frac{i\omega t}{\cancel{\hbar}}\right)^3 \underbrace{[S_z, [\quad]]}_{=\cancel{i\hbar^3} S_y} + \cdots$$

$$= S_x\left[1 - \frac{1}{2!}(\omega t)^2 + \cdots\right] - S_y\left[(\omega t) - \frac{1}{3!}(\omega t)^3 + \cdots\right]$$

$$= S_x \cos\omega t - S_y \sin\omega t$$

$$\Rightarrow \quad \langle S_x\rangle_t = \langle S_x\rangle_0 \cos\omega t - \langle S_y\rangle_0 \sin\omega t$$

$$\langle S_y\rangle_t = \langle S_y\rangle_0 \cos\omega t + \langle S_x\rangle_0 \sin\omega t$$

$$\langle S_z\rangle_t = \langle S_z\rangle_0 .$$

→ This is nothing but the rotation around $\hat{z}$-axis with angle $\phi = \omega t$!

But, there's a weird thing.

- $\langle \vec{S} \rangle_{2\pi} = \langle \vec{S} \rangle_0$; It's ok.

$$|\alpha, t\rangle = \mathcal{U}(t) \left[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| \right] |\alpha\rangle \qquad \| \ \phi = \omega t.$$

$$= e^{-\frac{i\phi}{2}} |\uparrow\rangle\langle\uparrow|\alpha\rangle + e^{\frac{i\phi}{2}} |\downarrow\rangle\langle\downarrow|\alpha\rangle$$

$$|\alpha, 2\pi\rangle = - |\alpha, 0\rangle. \quad !!!$$

The state comes back with a minus sign!

or.

↳ pressesion period $\tau = \frac{2\pi}{\omega}$ for $\langle \vec{S} \rangle$.

but $\tau_{\text{state ket}} = \text{"}\frac{4\pi}{\omega}\text{"}$ for $|\alpha\rangle$.

(3) Generalization : $SU(2)$ vs. $SO(3)$

- Pauli two-component formalism with the "Pauli" spinor.

ket

$$|\uparrow\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_\uparrow \quad , \quad |\downarrow\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_\downarrow$$

bra

$$\langle\downarrow| \doteq (1, 0) \equiv \chi_\uparrow^\dagger \quad , \quad \langle\downarrow| \doteq (0, 1) \equiv \chi_\downarrow^\dagger$$

a state

$$|\alpha\rangle \doteq \begin{pmatrix} \langle\uparrow|\alpha\rangle \\ \langle\downarrow|\alpha\rangle \end{pmatrix} \quad , \quad \langle\alpha| \doteq \left(\langle\alpha|\uparrow\rangle, \langle\alpha|\downarrow\rangle \right).$$